Int. J. Solids Structures Vol. 29, No. 17, pp. 2177–2180, 1992
Printed in Great Britain.

0020-7683/92 $5.00+ .00
Pergamon Press Ltd

A NOTE ON FINITE TWISTING AND TRANSVERSE BENDING OF SHEAR DEFORMABLE ORTHOTROPIC PLATES

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(*Received* 18 *July* 1991; *in revised form* 26 *November* 1991)

Abstract—In generalization of earlier results for non-shear deformable plates, this note obtains a system of linear ordinary differential equations for a problem of finite twisting and bending based on an intrinsic version of the equations for finite deflections of shear deformable plates. The differential equations of the system have constant coefficients for uniform plates, with this offering the possibility of an explicit determination of the effect of shear deformability on the stability characteristics of this non-linear structural problem.

INTRODUCTION

The recent reconsideration of the problem of combined twisting and bending based on an intrinsic form of the equations for small finite deflections of non-shear deformable plates (Reissner, 1992), suggests the possibility of an analogous analysis in which the effect of transverse shear deformability is taken into account. In what follows we undertake such an analysis through the use of a special case of the equations for finite deflections of shear deformable plates in Reissner (1986).

We begin by transforming the equations taken from Reissner (1986) into their intrinsic form. We then show that, in analogy to the earlier reduction of the two-dimensional eighth-order problem in Reissner (1992) to a fourth order ordinary differential equation problem, it is now possible, by a semi-inverse procedure, to effect a reduction of the tenth-order non-linear system of two-dimensional plate equations in Reissner (1986) to a *sixth* order system of ordinary differential equations. The same as for the limiting case of this result in the absence of shear deformability, this system turns out to be linear in the dependent variables but non-linear insofar as the relations between loads and deformations are concerned. It remains to complete our analysis by appropriate numerical evaluations.

THE DIFFERENTIAL EQUATIONS FOR SMALL FINITE DEFLECTIONS

The equilibrium and strain displacement equations are in accordance with Reissner (1986)

$$N_{x,x}+N_{t,y}=0, \quad N_{t,x}+N_{y,y}=0 \tag{1}$$

$$Q_{x,x}+Q_{y,y}+w_{,xx}N_x+2w_{,xy}N_t+w_{,yy}N_y=0 \tag{2}$$

$$M_{x,x}+M_{t,y}=Q_x, \quad M_{t,x}+M_{y,y}=Q_y \tag{3}$$

$$\varepsilon_x=u_{x,x}+\tfrac{1}{2}w_{,x}^2, \quad \varepsilon_y=u_{y,y}+\tfrac{1}{2}w_{,y}^2, \quad \varepsilon_t=u_{x,y}+u_{y,x}+w_{,x}w_{,y} \tag{4}$$

$$\gamma_x=\phi_x+w_{,x}, \quad \gamma_y=\phi_y+w_{,y} \tag{5}$$

$$\kappa_x=\phi_{x,x}, \quad \kappa_y=\phi_{y,y}, \quad \kappa_t=\phi_{x,y}+\phi_{y,x}. \tag{6}$$

In this N_x, N_y, N_t and Q_x, Q_y are mid-surface tangential and normal stress resultants, M_x, M_y, M_t are bending and twisting stress couples, ε_x, ε_y, ε_t, κ_x, κ_y, κ_t are stretching and bending strains, γ_x, γ_y are transverse shearing strains and u_x, u_y, w, ϕ_x, ϕ_y are translational and rotational displacement components. In as much as the strain measures ε, γ and κ are conjugates of the stress measures N, Q, M, in conjunction with the equilibrium equations (1)–(3), it is consistent to supplement the system (1)–(6) with constitutive relations of the form

$$\varepsilon_x = C_x N_x - C_v N_y, \quad \varepsilon_y = C_y N_y - C_v N_x, \quad \varepsilon_t = C_t N_t \tag{7}$$

$$\gamma_x = A_x Q_x, \quad \gamma_y = A_y Q_y \tag{8}$$

$$M_x = D_x \kappa_x + D_v \kappa_y, \quad M_y = D_y \kappa_y + D_v \kappa_x, \quad M_t = D_t \kappa_t. \tag{9}$$

We do not consider here a reduction of the system (1)–(9) to three simultaneous differential equations for the deflection w in conjunction with two stress functions K and J as in Reissner (1986). Instead we undertake a reduction of (1)–(6) to an *intrinsic* form, involving stress and strain measures only, as follows. We first deduce, as a consequence of (5) and (6), as expressions for the second derivatives of w,

$$w_{,xx} = \gamma_{x,x} - \kappa_x, \quad w_{,yy} = \gamma_{y,y} - \kappa_y, \quad 2w_{,xy} = \gamma_{x,y} + \gamma_{y,x} - \kappa_t. \tag{10}$$

From this we obtain two compatibility equations

$$2\kappa_{x,y} - \kappa_{t,x} = \gamma_{x,xy} - \gamma_{y,xx} \qquad 2\kappa_{y,x} - \kappa_{t,y} = \gamma_{y,xy} - \gamma_{x,yy}. \tag{11, 12}$$

A third compatibility equation follows from (4) in conjunction with (10) in the form

$$\varepsilon_{x,yy} - \varepsilon_{t,xy} + \varepsilon_{y,xx} = \tfrac{1}{4}(\kappa_t - \gamma_{x,y} - \gamma_{y,x})^2 - (\kappa_x - \gamma_{x,x})(\kappa_y - \gamma_{y,y}). \tag{13}$$

The intrinsic version of the equations of this plate theory is completed upon introducing (10) into (2) so as to have, in place of (2),

$$Q_{x,x} + Q_{y,y} = (\kappa_x - \gamma_{x,x})N_x + (\kappa_y - \gamma_{y,y})N_y - (\kappa_t - \gamma_{x,y} - \gamma_{y,x})N_t. \tag{14}$$

Equations (11)–(14) reduce to corresponding relations in Reissner (1992) upon setting $\gamma_x = \gamma_y = 0$, in conjunction with setting $A_x = A_y = 0$ in (8).

BOUNDARY CONDITIONS FOR TWISTING AND TRANSVERSE BENDING OF A RECTANGULAR PLATE

Given a plate with edges $x = \pm a$ and $y = \pm b$, we stipulate that the edges $y = \pm b$ are traction-free

$$y = \pm b, \quad N_t = N_y = Q_y = 0, \quad M_t = M_y = 0 \tag{15}$$

and that the edges $x = \pm a$ are acted upon by twisting moments T and transverse bending moments M.

With a base plane perpendicular stress resultant $Q_x + w_{,x} N_x + w_{,y} N_t$ we have as the expression for the twisting moment

$$T = \int_{-b}^{b} [(Q_x + w_{,x} N_x + w_{,y} N_t) y - M_t]_{x = \pm a}\, dy. \tag{16}$$

The corresponding transverse bending moment expression is

$$M = \int_{-b}^{b} [M_x + wN_x]_{x=\pm a}\,dy. \quad (17)$$

The global loading conditions (16) and (17) are here associated with three conditions of vanishing resultant edge force components, and one condition of the vanishing of the base-plane perpendicular resultant bending moment component. These four conditions may be written in the form

$$\int_{-b}^{b} [(1, y)N_x, N_t, Q_x + w_{,x}N_x + w_{,y}N_t]_{x=\pm a}\,dy = 0. \quad (18)$$

THE SEMI-INVERSE ONE-DIMENSIONAL SOLUTION

As in Reissner (1992) we assume that the constitutive coefficients in (7)–(9) are independent of x, and we utilize a semi-inverse procedure to obtain a rational class of solutions of the system (1), (3), (7)–(9) and (11)–(14), by stipulating that all measures of stress and strain are functions of y only.

We then have from (1) in conjunction with (15) that

$$N_t = N_y = 0 \quad (19)$$

and from (11) and (12) that

$$\kappa_x = k, \quad \kappa_t - \gamma_{x,y} = 2\theta \quad (20)$$

where k and θ are constants which remain to be determined.

The introduction of (19) and (20) into eqns (3), (13) and (14) leaves as a system of ordinary differential equations

$$M_t' = Q_x, \quad M_y' = Q_y, \quad Q_y' = kN_x \qquad \varepsilon_x'' = \theta^2 - (\kappa_y - \gamma_y')k \quad (21, 22)$$

with the primes indicating differentiation with respect to y. The above system is transformed, with the help of eqns (20) and (7)–(9), into a second order differential equation for M_t,

$$M_t - D_t(A_x M_t')' = 2D_t\theta \quad (23)$$

and into a fourth order equation for M_y

$$D_y[(C_x M_y'')'' - k^2(A_y M_y')'] + k^2 M_y = D_y k\theta^2 + D_v k^3. \quad (24)$$

The boundary conditions for these two equations are, on the basis of (15) and (21)

$$y = \pm b; \quad M_t = 0, \quad M_y = 0, \quad M_y' = 0. \quad (25)$$

When $A_x = A_y = 0$ eqns (23)–(25) reduce, upon some change in notation, to the corresponding results in Reissner (1992).

With M_t and M_y determined as functions of k and θ we may next obtain M and T from (17) and (18) in terms of k and θ. We find, upon certain transformations, with the help of (19)–(21)

$$M = \int_{-b}^{b} \left[\left(D_x - \frac{D_v^2}{D_y} \right) k + 2D_v \frac{M_y}{D_y} - \frac{M_y^2}{kD_y} - \frac{A_y(M_y')^2}{k^2} \right] dy \quad (26)$$

and

$$T = -\int_{-b}^{b}\left[2M_t + 2\theta\,\frac{M_y}{k}\right]\mathrm{d}y. \tag{27}$$

The fact that the one-dimensional results in (19)–(25) are consistent with the homogeneous global boundary conditions (18) is established in a manner which is analogous to the derivation of (26) and (27).

CONCLUDING REMARKS

Since the differential equations (23) and (24) have constant coefficients for plates with constant values of the constitutive coefficients, the determination of the functions $M(k,\theta)$ and $T(k,\theta)$, and the discussion of the associated stability problem, can be carried out in closed form in the same manner as was done in Reissner (1957) for the case with $A_x = A_y = 0$ and $D_y = D_X$, $D_v = \nu D_x$.

As was the case for the problem with $A_x = A_y = 0$ in Reissner (1992), it remains to be shown that the one-dimensional solution in accordance with (23)–(27) does in fact represent, for sufficiently large values of a/b, the interior portion of an asymptotic solution of a two-dimensional problem now with local boundary conditions

$$x = \pm a, \quad N_x = N_t = 0 \tag{28}$$

and

$$x = \pm a, \quad w = \mp\theta a y, \quad \phi_y = \pm\theta a, \quad \phi_x = \pm ka - \theta y \tag{29}$$

in place of the global conditions (16)–(18).

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